Name: Feedback

SA402 · Dynamic and Stochastic Models Fall 2022 · Uhan

Quiz 9 – 12/7/2022

Problem	Weight	Score
1a	1	
1b	1	
2a	1	
2b	1	
2c	1	
3	1	
4a	1	
4b	1	
Total		/ 80

Instructions. You have the entire class period to complete this quiz. You may use your plebe-issue calculator and the provided list of formulas for birth-death processes and standard queueing models. You may not use any other materials (e.g., notes, homework, books).

Show all your work. To receive full credit, your solutions must be completely correct, sufficiently justified, and easy to follow.

Problem 1. The Simplexville Post Office has 3 clerks on duty. Customers wait in a single queue and are served by the first available clerk, first-come first-served. The post office can only hold 30 customers total, including the 3 customers being served.

Potential customers arrive at the door of the post office a rate of 40 per hour. The interarrival times are exponentially distributed. When the post office has 20 or more customers total, potential customers balk with probability 1/2. In addition, any potential customers that arrive when the post office is full simply go elsewhere.

The average service time for a single clerk is 5 minutes per customer. The service times are exponentially distributed. Customers in the queue sometimes renege: the time a customer is willing to spend in the queue is exponentially distributed with a mean of 10 minutes.

Model this setting as a birth-death process by answering the following prompts.

a. Define the arrival rate in each state, in terms of the number of customers per hour.

See Case 2 in Lesson 12, as well as Problem 1 in the Lesson 12 Exercises for examples of how to model balking in the arrival rates of a birth-death process.

Note that when there are 20 to 29 customers in the system, a potential customer balks with constant probability 1/2. This is unlike the examples mentioned above, where the balking probability is an increasing function of the number of customers in the system.

b. Define the service rate in each state, in terms of the number of customers per hour.

See Case 5 in Lesson 12, as well as Problem 1a in the Review Problems for the Final Exam for examples of how to model reneging in the service rates of a birth-death process.

In this problem, we are told that the time a customer is willing to spend in the queue is exponentially distributed with a mean of 10 minutes. This means that each customer in the queue departs the system at a rate of 6 customers per hour. So, when there are 4 customers in the system (3 being served, 1 in the queue), the total service (departure) rate of the system is $12(3) + 6(1) = 42$ customers per hour.

How can you generalize this reasoning for when there are i customers in the system?

Problem 2. Sean and Dan's Pizza Shop has a single queue for waiting customers and 2 cashiers. One of the cashiers is on duty at all times. The other cashier goes on duty whenever the queue of customers becomes too long; in particular, when the shop has 4 or more customers (including the customers being served). The shop can hold at most 8 customers total.

Suppose that customer arrivals are well modeled as a Poisson process with a rate of 20 customers per hour. The cashiers both work at a rate of 15 customers per hour, and the service times are well modeled as exponential random variables.

This setting can be modeled as a birth-death process with the following arrival and service rates (in customers per hour):

$$\lambda_i = \begin{cases} 20 & \text{for } i = 0, 1, \dots, 7 \\ 0 & \text{for } i = 8, 9, \dots \end{cases} \qquad \mu_i = \begin{cases} 15 & \text{for } i = 1, 2, 3 \\ 30 & \text{for } i = 4, 5, \dots \end{cases}$$

The steady-state probabilities are:

$$\pi_0 = 0.09 \qquad \pi_1 = 0.13 \qquad \pi_2 = 0.17 \qquad \pi_3 = 0.22 \qquad \pi_4 = 0.15$$
$$\pi_5 = 0.10 \qquad \pi_6 = 0.07 \qquad \pi_7 = 0.04 \qquad \pi_8 = 0.03$$

a. Over the long run, what fraction of time are both cashiers busy?

Note that both cashiers are busy when the system has 4 or more customers. Recall that π_i is the long-run fraction of time that the system has exactly i customers.

b. Over the long run, what is the expected number of customers in the shop?

Note that this is system is <u>not</u> an M/M/$s = 2$ queue, because the queue does <u>not</u> have infinite capacity. Therefore, you cannot use the equations for M/M/s queues we covered in Lesson 14. Instead, you need to resort to the equations for performance measures for birth-death processes that we covered in Lesson 12.

See Problem 1b in the Review Problems for the Final Exam for a similar example.

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c. Over the long run, what is the expected time a customer spends in the shop?

See the feedback for Problem 2b.

Also, note that $\lambda_8 = 0$ and $\pi_8 = 0.03$. As a result, λ_{eff} will be less than 20.

See Problem 1c in the Review Problems for the Final Exam for a similar example.

Problem 3. Four Guys Burgers and Fries has 4 cashiers at its Simplexville location. Customers wait in a single queue and are served by the first available cashier, first-come first-served. The average service time is 2 minutes per customer, and customers arrive at a rate of 24 per hour. The interarrival times and the service times are best modeled with the exponential distribution. The Simplexville location is enormous and popular, so for all intents and purposes, the restaurant has infinite capacity and has an infinite number of possible customers.

Which standard queueing model fits this setting best?

See page 1 of Lesson 14 to remind yourself of the standard queueing notation, and then identify the 6 different pieces (arrival process, service process, s, n, k, queueing discipline) for this problem.

See Problem 1a in the Lesson 14 Exercises and Problem 2a in the Review Problems for the Final Exam for similar examples.

Problem 4. You have been asked to take over the task of staffing maintenance teams at a facility that repairs and maintains SH-60 Seahawk helicopters. The helicopters arrive at the maintenance facility at a rate of 6 per week, and are processed on a first-come-first-served basis. It takes a maintenance team 1 day on average to repair 1 helicopter. The facility currently has 1 maintenance team working at any given time.

Model this system as an M/M/1 queue. For the problems below, use weeks as your time unit.

a. Over the long run, what is the fraction of time that there is a repair backlog (i.e., at least 1 helicopter awaiting repair)?

Note that if there is at least 1 helicopter awaiting repair, then there must be 2 or more helicopters in the system: 1 helicopter being repaired, and the other helicopters awaiting repair. (If you interpreted "1 helicopter awaiting repair" as "a helicopter being repaired or waiting to be repaired", I accepted that as well.)

Therefore, we want the long-run fraction of time there are at least 2 helicopters in the system. Like in Problem 2a, recall that π_i is the long-run fraction of time that the system has exactly i helicopters.

Many of you computed π_0 and π_1, but did not use them to answer the question.

b. Suppose you find that $\pi_0 = 0.14$ (this may or may not match what you found in part a.). What is the expected time in queue (i.e., expected delay) for a helicopter that comes in for repair?

See Example 4 in Lesson 14, Problem 1d in the Lesson 14 Exercises, and Problem 2cd in the Review Problems for the Final Exam for similar examples.

Note that the list of useful formulas has the expected number of customers in queue for an M/M/s queue, $\ell_q = \dfrac{\pi_s \rho}{(1-\rho)^2}$, but does not explicitly have the accompanying expected delay.

However, Little's law still applies here! If we model an M/M/s queue with arrival rate λ as a birth-death process, the arrival rate when there are i customers in the system is $\lambda_i = \lambda$ for $i = 0, 1, 2, \dots$ (see Example 2 of Lesson 14). Therefore,

$$\lambda_{\text{eff}} = \sum_{i=0}^{\infty} \lambda_i \pi_i = \sum_{i=0}^{\infty} \lambda \pi_i = \lambda \underbrace{\sum_{i=0}^{\infty} \pi_i}_{\substack{=1\text{, since all the}\\ \text{steady-state}\\ \text{probabilities}\\ \text{must sum to 1}}} = \lambda,$$

and so $w_q = \dfrac{\ell_q}{\lambda_{\text{eff}}} = \dfrac{\ell_q}{\lambda}$. Note this matches the equation given on page 3 of Lesson 14.

tl;dr: For M/M/s and M/M/∞, the effective arrival rate λ_{eff} is equal to the arrival rate λ. Little's law still applies.